

On the Vanishing Topology of Isolated Cohen-Macaulay Codimension 2 Singularities

Anne Fröhbis–Krüger, Matthias Zach

Institut f. Alg. Geometrie, Leibniz Universität Hannover, Germany

November 10, 2016

Abstract

The topology of isolated complete intersections is well studied, but beyond this class not much is known. Isolated Cohen-Macaulay codimension 2 singularities share many common features with isolated complete intersection singularities, but they also exhibit some striking new behaviour. One such instance was observed by Damon and Pike [9] in their study of the vanishing topology and Euler characteristic, where they took this class of singularities as examples. In this article, we explore the background and geometrical meaning of their findings by determining the Betti numbers explicitly and explain the new phenomena. An important tool here is the Tjurina modification relating a Cohen-Macaulay codimension 2 singularity to a finite number of complete intersection singularities.

1 Introduction

Isolated hypersurface singularities and, a bit more generally, isolated complete intersection singularities have been a central focus of singularity theory ever since the famous A-D-E list of Arnold [1] classifying the simple hypersurface singularities. Classification questions as well as topological and analytic properties of these singularities have been studied intensively over the past decades (e.g. [14], [16]); many of the properties can be expressed in terms of invariants of the singularities and relations among these. The most famous ones are the Milnor number μ on the topological side and the Tjurina number τ , i.e. the dimension of the T^1 , related to the first order deformations. Following Damon and Pike we will view μ as the “defect of the Euler characteristic”. For an isolated singularity with unique Milnor fiber this is the difference between the topological Euler characteristic of any smooth fibre and the central fiber in a deformation. For quasihomogeneous ICIS, equality of the two numbers holds (see [15]); for any general ICIS we still have $\mu \geq \tau$ (see [19]).

As soon as we pass beyond ICIS, however, only a few results are known. The easiest non-ICIS case is the case of isolated Cohen-Macaulay codimension

2 singularities where the Hilbert-Burch theorem allows a description by means of the presentation matrix of the vanishing ideal. This case has recently come into focus of ongoing research, starting with the classification simple singularities in this case, first for space curves in [11] and later for arbitrary dimension in [10]. This, in turn led to further study of the properties of these singularities as in [25], [22] and most recently in [13] and [6]. But up to now the properties of the Milnor fibre of such singularities are far from being explored in all details.

Damon and Pike studied the vanishing topology of a generalization of the Milnor fibre, the so-called singular Milnor fibre in [9], in a much broader context. They used the simple isolated Cohen-Macaulay codimension 2 singularities from the list by Fröhbis-Krüger and Neumer [10] as examples to illustrate their methods. For the surface case, their approach (as well as independently a different approach introduced by da Silva Pereira and Ruas in [25]) provided direct computations for the Milnor number, which is the second (and only non-vanishing) Betti number of the Milnor fiber in this dimension.

Moving one dimension higher to isolated Cohen-Macaulay codimension 2 singularities of dimension 3, the situation becomes more delicate. Some facts are still known: the existence of a smoothing and the vanishing of the first Betti number have been shown by Greuel and Steenbrink in [16], whereas the vanishing of homology in degrees bigger than the complex dimension of the underlying variety is a well-known consequence of the Lefschetz hyperplane theorem (see [21]). The methods of Damon and Pike then allowed the computation of the difference $b_3 - b_2$ of the two remaining Betti-numbers, but not of each of the two separately. This is the defect of the Euler characteristic which we understand as a generalization of the Milnor number μ as mentioned earlier. Nevertheless their results provided striking evidence for b_2 to be nonzero in some of the families of simple threefold singularities from [10], as the computed value of the difference was negative, but never smaller than -1 for these.

In this article, we extend a technique which was previously only used for surfaces, the Tjurina modification (see e.g. [27] or [26]), applying it to Cohen-Macaulay codimension 2 singularities in general. It allows us to relate the given singularity or family of singularities to a local complete intersection scheme or a family of local complete intersections factoring through a given deformation. Using this tool, we are then able to explain the observation of Damon and Pike, explicitly compute that Betti numbers b_2 and b_3 for all simple isolated Cohen-Macaulay codimension 2 threefold singularities and even state a large class of such singularities (including the simple ones) for which b_2 has to have the value 1.

In section 2 we briefly recall those of the known results about isolated Cohen-Macaulay codimension 2 singularities which will be needed later on. In the section 3, we consider the notion of a Tjurina modification in detail and extend it suitably to higher dimensions, larger matrices and families of singularities.

In these two sections, we also give very explicit descriptions of the objects and statements to allow the use of the results in algorithmic and experimental approaches to problems of similar flavour. We then recall important facts about the Milnor fibre and prove the main results in section 4. The last section contains the application of the results to explicit examples.

We would like to thank Terence Gaffney, Wolfgang Ebeling, Wim Veys, Slawomir Rams, Victor Gonzalez Alonso, Miguel Marco and Jesse Kass for fruitful exchange of ideas on the topics related to this article. This work is partially supported by funds of the research project 'Experimental methods in Computer Algebra' of the NTH.

2 Basic Facts on isolated Cohen-Macaulay codimension 2 singularities

Isolated Cohen-Macaulay codimension 2 singularities (abbreviated by ICMC2 in the following) provide the most accessible setting for non-complete-intersection singularities. Contact equivalence, semi-universal deformation and simple objects are known in this case, see [11] and [10]. For the list of simple objects in the dimension 3 case, see table 1. In this section, we will briefly recall some of the basic facts for reader's convenience.

Using the Hilbert-Burch theorem, all Cohen-Macaulay germs of codimension 2 can be expressed as the maximal minors of $(t + 1) \times t$ -matrices M and vice versa. In the same way, flat deformations can be represented by perturbations of the matrix M and any perturbation gives rise to a flat deformation (cf. Burch [5], Schaps [24]). The minimal matrix size t is called the Cohen-Macaulay type of the singularity.

Classification up to contact-equivalence means that two singularities are considered equivalent, if their germs are isomorphic. The action of the contact-group translates directly to the application of coordinate changes and row and column operations on M . A singularity is called simple, if it can only deform into finitely many different equivalence classes (types) of singularities.

For a more consistent notation, we prefer to describe the Cohen-Macaulay codimension 2 singularities by their presentation matrix instead of the vanishing ideal. This requires a reformulation of $T_{X,0}^1$ in terms of the presentation matrix:

Lemma 2.1 ([11]). $T_{X,0}^1$ is given by

$$T_{X,0}^1 \cong \text{Mat}(t + 1, t; \mathbb{C}\{x_1, \dots, x_n\}) / (J_M + \text{Im}(g))$$

where J_M is the submodule generated by the matrices of the form

$$\begin{pmatrix} \frac{\partial M_{11}}{\partial x_j} & \cdots & \frac{\partial M_{1t}}{\partial x_j} \\ \vdots & & \vdots \\ \frac{\partial M_{(t+1)1}}{\partial x_j} & \cdots & \frac{\partial M_{(t+1)t}}{\partial x_j} \end{pmatrix} \quad \forall 1 \leq j \leq m$$

and g is the map

$$\begin{aligned} \text{Mat}(t+1, t+1; \mathbb{C}\{x_1, \dots, x_m\}) \oplus \text{Mat}(t, t; \mathbb{C}\{x_1, \dots, x_m\}) \\ \xrightarrow{g} \text{Mat}(t+1, t; \mathbb{C}\{x_1, \dots, x_m\}) \end{aligned}$$

mapping $(A, B) \mapsto AM + MB$.

It is a well-known fact that $T_{X,0}^2 = 0$ for Cohen-Macaulay codimension 2 singularities, i.e. that there are no obstructions to lifting first order deformations. As the Cohen-Macaulay codimension 2 singularities, we are considering, are isolated, $T_{X,0}^1$ is of finite dimension $\dim_{\mathbb{C}} T_{X,0}^1 = \tau$. Hence the base of the semiuniversal deformation of $(X, 0)$ is \mathbb{C}^τ and the total space is given by the minors of the matrix

$$M_{su} = M + \sum_{i=1}^{\tau} s_i m_i \in \text{Mat}(t+1, t; \mathbb{C}[s_1, \dots, s_\tau]\{\underline{x}\})$$

where the s_i are the coordinates of \mathbb{C}^τ and $\{m_1, \dots, m_\tau\}$ is a \mathbb{C} -basis of $T^1(X, 0)$ in the matrix notation of Lemma 1.

The above description of $T_{X,0}^1$ in terms of the presentation matrix, the non-existence of obstructions and the explicit description of the semiuniversal deformation are the main reasons why this class of singularities is a natural choice for a first step beyond isolated complete intersection singularities: To study their deformations we can follow the main ideas used in the complete intersection case. This led to the complete classification of simple isolated Cohen-Macaulay codimension 2 singularities found in [10] which lists nearly 30 series and more than 20 exceptional cases in dimensions $0 \leq \dim(X, 0) \leq 4$.

On the other hand, there are certain structural properties of our singularities which do not coincide with the complete intersection case and are based on the fact that the ring of $(X, 0)$ is a determinantal ring (see e.g. [3] for a textbook on determinantal rings). A first occurrence of this situation is linked to the following well-known fact:

Lemma 2.2. For $k \leq l$, let $G \in \text{Mat}(l, k; \mathbb{C}\{y_{1,1}, \dots, y_{l,k}\})$ be the matrix with entries $g_{i,j} = y_{i,j}$. Denote by $V_r \subset \mathbb{C}^{l \cdot k} = \text{Mat}(l, k; \mathbb{C})$ the variety of matrices with rank $\leq k - r$. These form a chain

$$\mathbb{C}^{l \cdot k} = V_0 \supset V_1 \supset \cdots \supset V_k = \{0\},$$

where the variety V_r is defined by the ideal generated by all $k - r + 1$ -minors of G . For $k > r > 0$ the singular locus of V_r is precisely V_{r+1} .

In fact, one implication of the last statement is obvious: the entries of the jacobian matrix of the ideal of r -minors of G are $\mathbb{C}\{\underline{y}\}$ -linear combinations of the $(r - 1)$ -minors as we can easily check by direct computation.

Now suppose an ICMC2 singularity $(X, 0) \subset (\mathbb{C}^n, 0)$ is given by a matrix $M \in \text{Mat}(t + 1, t; \mathbb{C}\{\underline{x}\})$. We can regard this matrix as a map

$$M : (\mathbb{C}^n, 0) \rightarrow \text{Mat}(t + 1, t; \mathbb{C}) \cong \mathbb{C}^{t \cdot (t+1)},$$

which by abuse of notation we also denote by M . Then the singularity $(X, 0)$ appears as the preimage $M^{-1}(V_1)$.

Corollary 2.3. Let $M \in \text{Mat}(t+1, t; \mathbb{C}\{x_1, \dots, x_n\})$ a presentation matrix of an isolated Cohen-Macaulay codimension 2 singularity $(X, 0)$ of Cohen-Macaulay type t . Then the locus defined by the $(t - 1)$ -minors of M is either the origin or empty.

Proof. As $(X, 0)$ is a germ of an isolated singularity, the singular locus is the origin. Regarding M as a map to $\mathbb{C}^{t \cdot (t+1)}$ gives a ring homomorphism

$$\begin{aligned} M^* : \mathbb{C}\{y_{i,j} \mid 1 \leq i \leq t+1, 1 \leq j \leq t\} &\longrightarrow \mathbb{C}\{x_1, \dots, x_n\} \\ y_{i,j} &\longmapsto m_{i,j}. \end{aligned}$$

Here $m_{i,j} \in \mathbb{C}\{x_1, \dots, x_n\}$ denotes the entry of the matrix M in the i -th row and j -th column. Let $\langle \delta_1, \dots, \delta_{t+1} \rangle$ be the ideal generated by the $t + 1$ maximal minors $\delta_i \in \mathbb{C}\{y\}$ of the matrix G from the previous lemma, i.e. the vanishing ideal of $V_1 \subset \mathbb{C}^{t \cdot (t+1)}$. Then the ideal $I \subset \mathbb{C}\{\underline{x}\}$ defining X is given by

$$f_i(\underline{x}) = \delta_i(M(\underline{x})).$$

Hence the jacobian matrix factors

$$J_f(\underline{x}) = J_g(M(\underline{x})) \cdot J_M(\underline{x}).$$

Now by the preceding lemma the matrix $J_g(M(\underline{x}))$ has entries contained in the ideal of $(t - 1)$ -minors of M and hence does $J_f(\underline{x})$ and all ideals of minors thereof. The statement follows immediately from this inclusion of ideals. \square

Remark 2.4. Of course, Corollary (2.3) can also be proved more directly. However, that does not illustrate the point in question. This is the more elegant argument: The presentation matrix M of the isolated singularity $(X, 0)$ describes the relations of the generators of the conormal module I/I^2 . Thus the ideal of $t - 1$ -minors is the second fitting ideal $\text{Fitt}_2(I/I^2)$ of the conormal module. Its vanishing locus is the set of primes where I/I^2 cannot be generated by 2 elements. But X is of codimension 2 and I/I^2 is locally free on the smooth part. Hence

$$V(\text{Fitt}_2(I/I^2)) \subset \text{Sing}(X) = \{0\}.$$

Remark 2.5. Regarding the matrix M of an ICMC2 singularity as a map to the space of matrices as used in Corollary (2.3) provides a different perspective to those of the properties of our singularities which originate from the determinantal structure: Cohen-Macaulay codimension 2 singularities are among the classes of singularities for which deformations of the space germs coincide with deformations of M as a map (see [4], chapter 4 and 5). The appropriate notion of equivalence in this context is \mathcal{K}_V -equivalence (see [7], [8]), but we will not need this notion in our considerations.

In this article, the interplay of both aspects, i.e. of the similarities to the ICIS case and of the structural properties of determinantal singularities, will be essential to studying the topology of the singularities in question. More precisely, we shall even see contributions of both kinds in the topology.

3 Tjurina modifications revisited

Central to our considerations will be a not so widely known tool that was developed by G. Tjurina in [27] for her study of rational triple point singularities. After finding that such surface singularities can be realized by a system of 3 equations, which we can easily recognize as 2-minors of a 2×3 matrix, she considers the map to \mathbb{P}^1 which maps each point of the rank-1-locus of the matrix to the corresponding (non-zero) column vector of the matrix. Resolving the locus of indeterminacy of this map (i.e. the rank-0-locus) then provides her with a local complete intersection which only possesses rational double point singularities.

This construction has later also been used in the thesis of D. van Straten [26], where its name was coined, and in a few other articles. However, it has – to our knowledge – never been applied beyond the case of surface singularities of Cohen-Macaulay-type $t = 2$. As we shall apply it to the 3-dimensional case and as we do not want to restrict our methods to the case of Cohen-Macaulay-type $t = 2$, we will generalize Tjurina’s construction here.

Construction 3.1. (Tjurina modification for generic determinantal varieties)

Consider the varieties $V_r \subset \mathbb{C}^{l \times k}$, $k \leq l$, as in Lemma 2.2. For a general point, i.e. a general $l \times k$ -matrix $A \in V_r$, the row vectors of A span a $k - r$ dimensional hyperplane $P_A \subset \mathbb{C}^k$. This determines a rational map to the Grassmannian of $(k - r)$ -planes in k -space.

$$\begin{aligned} P : V_r &\dashrightarrow \text{Grass}(k - r, k) \\ A &\mapsto P_A. \end{aligned}$$

Clearly P is defined on the open set $V_r \setminus V_{r+1}$. Regarding $\text{Grass}(k - r, k) \subset \mathbb{P}(\bigwedge^{k-r} \mathbb{C}^k)$ as a subvariety of projective space it becomes clear that P can

always be expressed in terms of $k - r$ -minors of A . As a projective variety the Grassmannian is complete and we can blow up the rational map P to obtain

$$W_r := \overline{\Gamma_P(V_r \setminus V_{r+1})} \subset \mathbb{C}^{l \cdot k} \times \text{Grass}(k - r, k)$$

as the closure of the graph of P restricted to $V_r \setminus V_{r+1}$ together with the canonical projection π and the prolongation \hat{P}

$$\begin{array}{ccc} W_r & & \\ \downarrow \pi & \searrow \hat{P} & \\ V_r & \xrightarrow{P} & \text{Grass}(k - r, k) \end{array}$$

In particular π is a resolution of the singularities of V_r .

Remark 3.2. For calculating W_r explicitly, we cover the projective variety $\text{Grass}(k - r, k) \subset \mathbb{P}(\bigwedge^{k-r} \mathbb{C}^k)$ by the standard affine charts. Similarly to writing a point $p \in \mathbb{P}^n$ in projective n -space as $p = (s_0 : \dots : s_n)$ in projective coordinates and thus also indicating the line $L(p) = \text{span}((s_0, \dots, s_n)^T) \subset \mathbb{C}^{n+1}$ sitting over p in the tautological bundle, we write a point $z \in \text{Grass}(k - r, k)$ as a $(k - r) \times k$ -matrix B . The standard cover is indexed by subsets $\alpha \subset \{1, \dots, k\}$ of cardinality $\#\alpha = k - r$. Analogous to normalizing the projective coordinates of a point $p = (s_0 : \dots : s_i : \dots : s_n)$ in the i -th chart of projective space to

$$p = \left(\frac{s_0}{s_i} : \dots : 1 : \dots : \frac{s_n}{s_i} \right) = \left(s_0^{(i)} : \dots : 1 : \dots : s_n^{(i)} \right),$$

we require the maximal square submatrix of B indexed by α to be the unit matrix. Thus we obtain affine coordinates $(z_{i,j}^{(\alpha)})_{i,j}$. For example if $\alpha = \{1, \dots, k - r\}$ we write a point $z \in U_\alpha \subset \text{Grass}(k - r, k)$ as

$$B_\alpha(z) = \begin{pmatrix} 1 & 0 & \dots & 0 & z_{1,k-r+1}^{(\alpha)} & \dots & z_{1,k}^{(\alpha)} \\ 0 & 1 & \ddots & \vdots & z_{2,k-r+1}^{(\alpha)} & \dots & z_{2,k}^{(\alpha)} \\ \vdots & \ddots & \ddots & 0 & \vdots & & \vdots \\ 0 & \dots & 0 & 1 & z_{k-r,k-r+1}^{(\alpha)} & \dots & z_{k-r,k}^{(\alpha)} \end{pmatrix}$$

The subspace $L(z) \subset \mathbb{C}^k$ sitting over z is now given by the span of the rows of the above matrix. Given a generic $l \times k$ matrix $G \in V_r \subset \mathbb{C}^{l \cdot k}$, requiring the span of the rows of G to be contained in $L(z)$ therefore amounts to asking for the $k - r + 1$ -minors of the matrix

$$\begin{pmatrix} B_\alpha \\ G \end{pmatrix}$$

to vanish. In fact the variety $W_r \subset \mathbb{C}^{l \cdot k} \times \text{Grass}(k - r, k)$ which is locally defined by these minors is already the strict transform of V_r under the blowup of P in our construction.

In the setting of ICMC2 singularities we will only be concerned with $(t + 1) \times t$ -matrices of rank $t - 1$. A $t - 1$ -dimensional subspace L in \mathbb{C}^t is uniquely determined by the class of a normal vector $[\vec{n}_L] \in \mathbb{P}^{t-1} \cong \text{Grass}(t - 1, t)$. The identification of $\text{Grass}(t - 1, t)$ with \mathbb{P}^{t-1} is given on the standard cover by identifying a point¹

$$(s_1^{(i)} : \dots : s_{i-1}^{(i)} : 1 : s_{i+1}^{(i)} : \dots : s_t^{(i)})$$

in the chart $\{s_i \neq 0\}$ in \mathbb{P}^{t-1} with the matrix

$$B_i(s) = \begin{pmatrix} 1 & 0 & \dots & 0 & -s_1^{(i)} & 0 & \dots & \dots & 0 \\ 0 & 1 & \ddots & \vdots & \vdots & \vdots & & & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \vdots & & & \vdots \\ 0 & \dots & 0 & 1 & -s_{i-1}^{(i)} & 0 & \dots & \dots & 0 \\ 0 & \dots & \dots & 0 & -s_{i+1}^{(i)} & 1 & 0 & \dots & 0 \\ \vdots & & & \vdots & \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & & & \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & -s_t^{(i)} & 0 & \dots & 0 & 1 \end{pmatrix} \quad (1)$$

The equations of the Tjurina transform W_1 of $V_1 \subset \mathbb{C}^{t \cdot (t+1)}$ therefore take a particular simple form: For a point $s = (s_1 : \dots : s_t) \in \mathbb{P}^{t-1}$ we just require the vector $\vec{s} = (s_1, \dots, s_t)^T$ to be perpendicular to the columns of a matrix $G \in V_1$.

Corollary 3.3. Let $G = (y_{i,j})_{i,j} \subset \mathbb{C}\{y\}$ be the generic $(t + 1) \times t$ matrix and $(s_1 : \dots : s_t)$ the projective coordinates of $\mathbb{P}^{t-1} = \text{Grass}(t - 1, t)$. The Tjurina transform $W_1 \subset \mathbb{C}^{(t+1)t} \times \mathbb{P}^{t-1}$ of V_1 is the zero locus of the equations

$$\begin{pmatrix} y_{1,1} & \dots & y_{1,t} \\ \vdots & & \vdots \\ y_{t+1,1} & \dots & y_{t+1,t} \end{pmatrix} \cdot \begin{pmatrix} s_1 \\ \vdots \\ s_t \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (2)$$

Proof. Computing each chart with the matrix B_i as in (1) gives the local equations. But these can be easily recognized as the dehomogenization of the equations given by (2). \square

From now on we will denote a representative of an isolated singularity by X_0 , i.e. with an additional index 0. This is due to consideration of deformations in the sequel, where the singularity $(X_0, 0)$ is embedded as the special fiber in a total space $(X, 0) \xrightarrow{\varepsilon} (B, 0)$ fibered over some base $(B, 0)$ by deformation parameters ε . Consequently for a choice of representatives the fiber over any nonzero $\varepsilon \in B$ is denoted by X_ε .

¹The non-standard numbering was chosen for consistency with the numbering of matrix entries.

Construction 3.4. (Tjurina modification for ICMC2 singularities) As pointed out in remark 2.5 an ICMC2 singularity $(X_0, 0) \subseteq (\mathbb{C}^m, 0)$ can be studied by means of a polynomial map $M : U \rightarrow \mathbb{C}^{(t+1)t}$ for a chosen representative $X_0 \subset U$ of $(X_0, 0)$ in a neighborhood U of the origin ². Concatenation with $P : V_1 \dashrightarrow \mathbb{P}^{t-1}$ gives a rational map

$$P \circ M : X_0 \dashrightarrow \mathbb{P}^{t-1}.$$

Because $P \circ M$ is expressed in the projective coordinates of \mathbb{P}^{t-1} in terms of $(t-1)$ minors of M , it is well defined outside the singular locus of X by Corollary 2.3.

Now we define the Tjurina modification Y_0 to be the fiber product $X_0 \times_{V_1} W_1$ in the following diagram:

$$\begin{array}{ccc} X_0 \times_{V_1} W_1 & \xrightarrow{\hat{M}} & W_1 \\ \pi \downarrow & & \downarrow \rho \\ X_0 & \xrightarrow{M} & V_1 \end{array} \quad \begin{array}{c} \searrow \hat{P} \\ \xrightarrow{P} \end{array} \mathbb{P}^r \quad (3)$$

On the level of equations this means nothing but regarding M as a matrix with polynomial entries and requiring the equations of the system

$$M \cdot \begin{pmatrix} s_1 \\ \vdots \\ s_t \end{pmatrix} = 0$$

to hold in $U \times \mathbb{P}^{t-1}$. Clearly outside the singular locus $\{0\}$ the map $\pi : Y_0 \rightarrow X_0$ is an isomorphism, while the origin itself is substituted with the whole Grassmannian \mathbb{P}^{t-1} .

Example 3.5. Let us consider the (non-simple) ICMC2 singularity $(X_0, 0) \subset (\mathbb{C}^5, 0)$ given by the 3-minors of the matrix

$$M = \begin{pmatrix} x & y - v & y + z \\ y & z - v & x + u \\ z & 0 & x - u \\ 0 & u & v \end{pmatrix}.$$

Let $(s_1 : s_2 : s_3)$ be the projective coordinates of $\mathbb{P}^2 = \text{Grass}(2, 3)$. Then we obtain $Y_0 \subset \mathbb{C}^5 \times \mathbb{P}^2$ as the zero locus of the equations

$$\begin{pmatrix} x & y - v & y + z \\ y & z - v & x + u \\ z & 0 & x - u \\ 0 & u & v \end{pmatrix} \cdot \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = 0. \quad (4)$$

²By abuse of notation we will also refer to such a representative as ICMC2.

The Tjurina transform Y_0 is still singular at 10 distinct points in $\{0\} \times \mathbb{P}^2 \subset \mathbb{C}^5 \times \mathbb{P}^2$. But there we only find 3-dimensional A_1 singularities embedded in higher dimensional space. Thus the situation became much simpler. Consider e.g. the singularity at the point $p = (0, (1 : 0 : 0))$ in the chart $s_1 \neq 0$: The first three lines of the system (4) define a smooth variety H of dimension 4 around p . Inside H the equation

$$s_2^{(1)} \cdot u + s_3^{(1)} \cdot v = 0$$

in the last line provides the A_1 singularity.

Any deformation of an ICMC2 singularity X_0 is described by a perturbation of the entries of the matrix M defining X_0 . As the process of Tjurina modification is based on the determinantal structure, it can be applied to all fibers of such a family simultaneously. Therefore it makes sense to ask whether (or in which situations) a Tjurina modification is well-behaved within the family.

Construction 3.6. (Tjurina modification in family) Let $X_0 \hookrightarrow X \xrightarrow{\varepsilon} \mathbb{C}$ be a deformation of an ICMC2 singularity $X_0 \subset \mathbb{C}^n$ of Cohen-Macaulay type t described by a matrix $M(\underline{x}, \varepsilon) \in \text{Mat}(t+1, t; \mathbb{C}\{\underline{x}, \varepsilon\})$. The Tjurina modification in family for this deformation is the result of applying the Tjurina modification to the total space $X \xrightarrow{\varepsilon} \mathbb{C}$ which leads to a diagram extending diagram (3) above:

$$\begin{array}{ccccccc} X_0 \times_{V_1} W_1 & \hookrightarrow & X \times_{V_1} W_1 & \xrightarrow{\hat{M}} & W_1 & & \\ \pi_0 \downarrow & & \pi \downarrow & & \rho \downarrow & \searrow \hat{P} & \\ X_0 & \hookrightarrow & X & \xrightarrow{M} & V_1 & \xrightarrow{P} & \mathbb{P}^r \\ \downarrow & & \varepsilon \downarrow & & & & \\ \{0\} & \longrightarrow & \mathbb{C} & & & & \end{array} \quad (5)$$

The equations defining $Y = X \times_{V_1} W_1$ in $\mathbb{C}^n \times \mathbb{C} \times \mathbb{P}^{t-1}$ are again

$$M(\underline{x}, \varepsilon) \cdot \vec{s} = 0$$

with $\vec{s} = (s_1, \dots, s_t)^T$ the vector whose entries are the homogeneous coordinates of \mathbb{P}^{t-1} . For the special fiber $Y_0 = X_0 \times_{V_1} W_1$ one always obtains the same result as in the case of applying the Tjurina modification to the singularity alone by Construction.

Example 3.7. Consider the deformation with a parameter ε given by the matrix

$$M(\underline{x}, \varepsilon) = \begin{pmatrix} x & y - v & y + z + 2\varepsilon \\ y & z - v & x + u + 2\varepsilon \\ z & 3\varepsilon & x - u \\ 3\varepsilon & u & v \end{pmatrix}.$$

Let $X \subset \mathbb{C}^5 \times \mathbb{C}$ be the total space of the deformation

$$\begin{array}{ccc} X_0 & \longrightarrow & X \\ \downarrow & & \downarrow \varepsilon \\ \{0\} & \longrightarrow & \mathbb{C} \end{array}$$

The Tjurina modification in family in $\mathbb{C}^5 \times \mathbb{C} \times \mathbb{P}^2$ is now described by the equations

$$M(\underline{x}, \varepsilon) \cdot \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = 0$$

As a direct computation shows, all fibers of this family except the one over $\varepsilon = 0$ are smooth. This has an important consequence in the setting of Tjurina modification:

Proposition 3.8. If in the setting of Construction 3.6 above the deformation of X_0 over \mathbb{C} is a smoothing, the restriction of π to a smooth fiber

$$\pi_\varepsilon : Y_\varepsilon \rightarrow X_\varepsilon$$

in diagram (5) is an isomorphism.

Proof. For fixed ε the rational map

$$P \circ M(-, \varepsilon) : X_\varepsilon \dashrightarrow \mathbb{P}^{t-1}$$

is not well defined in the vanishing locus of $(t-1)$ -minors of M . By Corollary 2.3, this is contained in the singular locus of X_ε , which is empty for smooth fibers. Hence $P \circ M_\varepsilon$ is regular. \square

Unfortunately it is not at all clear that the families $Y \xrightarrow{\varepsilon \circ \pi} \mathbb{C}$ obtained by Tjurina modifications in family as in diagram (5) are flat. Whether or not this is the case, will in general depend on the deformation in question, the dimension of the singularity and the Cohen-Macaulay-type. Consider e.g. a space curve $X_0 \subset \mathbb{C}^3$ of Cohen-Macaulay-type 3 as the special fiber in a smoothing by a parameter ε , then the fiber Y_0 of the Tjurina transform over $\varepsilon = 0$ contains a \mathbb{P}^2 , while the other fibers stay 1-dimensional. This clearly contradicts flatness.

For simple ICMC2 singularities of dimension $\dim(X_0, 0) > 0$ this does not pose a problem³: All families in the classification of Neumer and Fröhbis-Krüger [10] have Cohen-Macaulay type $t = 2$. This will turn out to be sufficient to assure flatness in all of our cases of interest.

³We deliberately exclude the simple fat points here as their behaviour obviously differs from the higher dimensions, because any \mathbb{P}^k in the Y_0 would violate flatness.

Proposition 3.9. Let $(X_0, 0) \subset (\mathbb{C}^{n+2}, 0)$ be an ICMC2 singularity of dimension $n > 0$ and Cohen-Macaulay type $t \leq n + 1$. The Tjurina modification in family for a deformation $X_0 \hookrightarrow X \xrightarrow{\varepsilon} \mathbb{C}$

$$\begin{array}{ccc} Y_0 & \longrightarrow & Y \\ \pi_0 \downarrow & & \downarrow \pi \\ X_0 & \longrightarrow & X \\ \downarrow & & \downarrow \varepsilon \\ \{0\} & \longrightarrow & \mathbb{C} \end{array}$$

is flat over \mathbb{C} .

Proof. The Grassmannian in question is a \mathbb{P}^{t-1} . As usual let $(s_1 : \dots : s_t)$ be its projective coordinates and $M(\underline{x}, \varepsilon)$ the matrix describing the family X . The variety $Y_0 \subset \mathbb{C}^{n+2} \times \mathbb{P}^{t-1}$ is given by the $t + 1$ equations

$$M_0(\underline{x}) \cdot \begin{pmatrix} s_1 \\ \vdots \\ s_t \end{pmatrix} = 0$$

Now

$$\dim Y_0 = \max \{ \dim X_0, \dim \mathbb{P}^{t-1} \} = \max \{ n, t - 1 \} = \dim X_0,$$

because $\pi_0 : Y_0 \rightarrow X_0$ is an isomorphism on $Y_0 \setminus \pi_0^{-1}(\{0\})$ and the exceptional set $\pi_0^{-1}(\{0\})$ is a \mathbb{P}^{t-1} . Since X_0 had codimension 2 in \mathbb{C}^{n+2} we find Y_0 to have codimension $t + 1$ in $\mathbb{C}^{n+2} \times \mathbb{P}^{t-1}$. But locally in all charts, there are exactly $t + 1$ equations describing Y_0 . This means Y_0 is a locally complete intersection, so the induced deformation by $M(\underline{x}, \varepsilon)$ in the Tjurina modification in family is flat. \square

Remark 3.10. The above result was independently formulated by Jesse Kass for simple space curve singularities in his up-to-now unpublished work on Coxeter-Dynkin diagrams of space curve singularities.

An alternative way to check flatness of a family is checking the relation lifting property for the relations of generators of the defining ideal (cf. [2]).

Let $X_0 \subset \mathbb{C}^n$ be an ICMC2 singularity of Cohen-Macaulay type t and $M(\underline{x}, \varepsilon)$ be a matrix defining a deformation of X_0 over $(\mathbb{C}, 0)$. The ideal $J \subset \mathbb{C}\{\underline{x}, \varepsilon\}[s_1, \dots, s_t]$ defining the Tjurina transform $Y \subset \mathbb{C}^n \times \mathbb{C} \times \mathbb{P}^{t-1}$ is generated by the t equations $H_i(\underline{x}, \varepsilon, \underline{s}) = 0$ originating from the lines of the system

$$M(\underline{x}, \varepsilon) \cdot \begin{pmatrix} s_1 \\ \vdots \\ s_t \end{pmatrix} = 0.$$

The relation lifting property for flatness requires that any relation

$$\sum_j r_j \cdot h_j = 0$$

in $\mathbb{C}\{\underline{x}\}[\underline{s}]$ among the $h_j = H_j(\varepsilon = 0)$ can be lifted to a relation $\sum_j R_j \cdot H_j = 0$ in $\mathbb{C}\{\underline{x}, \varepsilon\}[\underline{s}]$ with $r_j = R_j(\varepsilon = 0)$. Now there is one relation among the H_j which comes naturally with a lifting:

Because the matrix M describes the syzygies of the generators of

$$I = \langle F_1(\underline{x}, \varepsilon), \dots, F_{t+1}(\underline{x}, \varepsilon) \rangle,$$

i.e. the ideal defining $X \subset \mathbb{C}^n \times \mathbb{C}$, we can write

$$0 = (F_1, \dots, F_{t+1}) \cdot M \cdot \begin{pmatrix} s_1 \\ \vdots \\ s_t \end{pmatrix} = (F_1, \dots, F_{t+1}) \cdot \begin{pmatrix} H_1 \\ \vdots \\ H_t \end{pmatrix}.$$

We call this the “relation by the maximal minors”. This leads to the following criterion for flatness of the Tjurina modification.

Lemma 3.11. If in the above setting the relations among the generators h_i of the ideal defining the Tjurina modification Y_0 of an ICMC2 singularity X_0 are generated by the Koszul relations and the relation by the maximal minors, then any Tjurina modification in family of a deformation of X_0 is again flat.

We finish the discussion of flatness by looking at one last example which does not satisfy the condition in the preceding criteria:

Example 3.12. The family of ICMC2 fat points defined by

$$\begin{pmatrix} 0 & x \\ x & y \\ y & \varepsilon \end{pmatrix}$$

does not give rise to a flat family by Tjurina modification: Tjurina modification provides

$$\hat{X}_\varepsilon = V(\underbrace{s_2 x}_{=f_1}, \underbrace{s_1 x + s_2 y}_{=f_2}, \underbrace{s_1 y}_{=f_3} + s_2 \varepsilon)$$

For $\varepsilon = 0$ we have the additional relation

$$s_1^2 f_1 - s_1 s_2 f_2 + s_2^2 f_3 = 0$$

among the generators of the ideal of Y_0 . The relation cannot be lifted to a relation of the whole family.

Considering this example from a geometric perspective, all fibers except the fiber at $\varepsilon = 0$ are zero-dimensional, but the special fiber additionally contains the \mathbb{P}^1 introduced by the Tjurina modification. As before such a jump in dimension clearly contradicts flatness.

To end this section, we want to study the effects of a Tjurina modification to versal families of ICMC2 singularities of Cohen-Macaulay type $t = 2$. These observations originate from direct computations, but will be useful for explicit examples:

Remark 3.13. Let $(X_0, 0) \subset (\mathbb{C}^n, 0)$ be an ICMC2 singularity at the origin with Cohen-Macaulay type $t = 2$ and $\dim X_0 > 0$. Let M be the corresponding presentation matrix. Expanding the matrix entries up to degree r and taking equivalence classes modulo $\langle x_1, \dots, x_n \rangle^{r+1}$, we can represent each such class by a matrix with polynomial entries of degree at most r . We shall refer to this representative as the r -jet of the presentation matrix, $j_r M$. More precisely, we need to prepare the subsequent discussion of the relationship of the deformations of X_0 and Y_0 and thus determine a very coarse classification of occurring 1-jets⁴.

As we are considering a germ around the origin, all entries of $j_1 M$ are homogeneous linear polynomials. We know that row and column operations on M leave the germ $(X_0, 0)$ unchanged, and we can safely pass to sufficiently general \mathbb{C} -linear combinations of the two original columns. The second column of $j_1 M$ thus holds up to 3 \mathbb{C} -linearly independent linear forms. By suitable row operations on M , we can then cancel linearly dependent entries of this column of $j_1 M$ and achieve that the zero entries are positioned below the non-zero entries. (Note that the sufficiently general linear combination of the columns now ensures that a row with a zero in the second entry also holds a zero in the first entry.) By an analytic change of coordinates, we can now choose the non-zero entries of the second column as new coordinates, starting with x_1 , and obtain the following four cases:

$$\begin{pmatrix} * & x_1 \\ * & x_2 \\ * & x_3 \end{pmatrix}, \quad \begin{pmatrix} * & x_1 \\ * & x_2 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} * & x_1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Here $*$ denotes an arbitrary entry.

Lemma 3.14. Let $(X_0, 0) \subset (\mathbb{C}^5, 0)$ be an ICMC2 threefold singularity of Cohen-Macaulay type $t = 2$. Then the Tjurina transform Y_0 has at most isolated singularities, iff X_0 is contact equivalent to a ICMC2 with presentation matrix

$$\begin{pmatrix} a & x_1 \\ b & x_2 \\ c & x_3 \end{pmatrix},$$

where $a, b, c \in \langle x_1, \dots, x_5 \rangle$.

Proof. Tjurina modification is an isomorphism outside the singular locus, which implies that the singular locus of Y_0 is contained in $E = \pi_0^{-1}(\{0\}) \cong \mathbb{P}^1$. Because E is irreducible, the singular locus of Y_0 is either a finite number of points or

⁴This list is, of course, loosely related to the lists of 1-jets in [10], but it contains significantly fewer classes, because here we are only hunting for a criterion for isolatedness of the singularities of the Tjurina transform.

the whole \mathbb{P}^1 .

If the matrix has the desired structure, we focus on one of the two standard affine charts of the exceptional \mathbb{P}^1 to show that there can be at most isolated singularities. As usual let $(s_1 : s_2)$ be the homogeneous coordinates of the exceptional curve \mathbb{P}^1 . The equations for $Y_0 \subset \mathbb{C}^5 \times \mathbb{P}^1$ in the chart $s_1 \neq 0$ are given by the ideal $I = \langle a + s_2x_1, b + s_2x_2, c + s_2x_3 \rangle$. The jacobian of this complete intersection reads

$$\begin{pmatrix} \frac{\partial a}{\partial x_1} + s_2 & \frac{\partial a}{\partial x_2} & \frac{\partial a}{\partial x_3} & \frac{\partial a}{\partial x_4} & \frac{\partial a}{\partial x_5} & x_1 \\ \frac{\partial b}{\partial x_1} & \frac{\partial b}{\partial x_2} + s_2 & \frac{\partial b}{\partial x_3} & \frac{\partial b}{\partial x_4} & \frac{\partial b}{\partial x_5} & x_2 \\ \frac{\partial c}{\partial x_1} & \frac{\partial c}{\partial x_2} & \frac{\partial c}{\partial x_3} + s_2 & \frac{\partial c}{\partial x_4} & \frac{\partial c}{\partial x_5} & x_3 \end{pmatrix}. \quad (6)$$

One of its 3-minors (first 3 columns) and hence of the ideal of the singular locus contains an element of the form $s_2^3 + \phi$ where the s_2 -degree of the remaining part ϕ is at most 2. This excludes the case of the singular locus being the whole exceptional curve E .

If, on the other hand, the matrix is not of the desired form, at least one row and hence at least one generator of I is contained in $\langle x_1, \dots, x_5 \rangle^2$, whence at least one row of the jacobian matrix – and thus the ideal of its 3-minors – is contained in $\langle x_1, \dots, x_5 \rangle$. Hence the singular locus would be 1-dimensional in this case. \square

In the case of the preceding lemma with only isolated singularities in the Tjurina transform, we now compare the infinitesimal deformations of the ICMC2 singularity $(X_0, 0)$ downstairs with those of the local complete intersection scheme $(Y_0, \mathbb{P}^1 \times \{0\})$ upstairs, where $\mathbb{P}^1 \times \{0\}$ is the exceptional locus of the Tjurina transform. For the affine germ $(X_0, 0)$ the first order deformations are encoded in the $\mathbb{C}\{\underline{x}\}$ -module $T_{X_0}^1$. The space of embedded first order deformations for the Tjurina transform $\iota : (Y_0, \mathbb{P}^1 \times \{0\}) \hookrightarrow \mathbb{P}^1 \times \mathbb{C}^5$ can be described as follows. Let \mathcal{I} be the ideal sheaf defining $(Y_0, \mathbb{P}^1 \times \{0\})$ in $(\mathbb{P}^1 \times \mathbb{C}^5, \mathbb{P}^1 \times \{0\})$. We take global sections of the normal bundle

$$N_{Y_0} = H^0(Y_0, \mathcal{H}om(\mathcal{I}, \mathcal{O}_{Y_0}))$$

and divide by those deformations coming from global sections of the tangent bundle $H^0(Y_0, \iota^*T_{\mathbb{P}^1 \times \mathbb{C}^5})$. The resulting quotient will be denoted by

$$N' := N_{Y_0} / H^0(Y_0, \iota^*T_{\mathbb{P}^1 \times \mathbb{C}^5}). \quad (7)$$

Note that the global section functor takes coherent sheaves to finitely generated $\mathbb{C}\{\underline{x}\}$ -modules. In fact N' is naturally a $\mathbb{C}\{\underline{x}\}$ -module with support in the point 0 and hence a finite dimensional vector space over \mathbb{C} . To see this observe that outside the singular locus $0 \in X_0$ (and outside $\mathbb{P}^1 \times \{0\} \subset Y_0$ respectively), the space Y_0 is described as a graph over X_0 and we therefore have a natural splitting of the normal bundle

$$N_{Y_0} = N_{X_0} \oplus T_{\mathbb{P}^1}|_{Y_0}.$$

Because the tangent bundle of \mathbb{P}^1 is globally generated, the second summand is killed when forming the quotient N' . But the first summand cancels on the smooth locus anyway.

It is clear from the construction that every deformation of $(X_0, 0)$ induces a deformation of $(Y_0, \mathbb{P}^1 \times \{0\})$. Let $(X_0, 0)$ be given by the matrix

$$M = \begin{pmatrix} a & x_1 \\ b & x_2 \\ c & x_3 \end{pmatrix} \in \text{Mat}(3, 2; \mathbb{C}\{x_1, \dots, x_5\}).$$

and let

$$H_1 = s_1 \cdot a + s_2 \cdot x_1, \quad H_2 = s_1 \cdot b + s_2 \cdot x_2, \quad H_3 = s_1 \cdot c + s_2 \cdot x_3 \in \mathbb{C}\{\underline{x}\}[s_1, s_2]$$

be the three equations defining the Tjurina transform Y_0 in $\mathbb{P}^1 \times \mathbb{C}^5$, which are homogeneous in \underline{s} . On the level of equations there is a map

$$\begin{array}{ccc} \{\text{Perturbations of } M\} & \xrightarrow{\Lambda} & \{\text{Perturbations of } \underline{H}\} \\ \text{\scriptsize 1:1} \Big\downarrow & & \Big\downarrow \text{\scriptsize 1:1} \\ \text{Mat}(3, 2; \mathbb{C}\{\underline{x}\}) & \xrightarrow[\substack{\Lambda \\ E_{i,j}^{(2,3)} \mapsto e_i s_j}]{} & ((\mathbb{C}\{\underline{x}\}[s_1, s_2])^3)_{(1)}, \end{array} \quad (8)$$

where the e_i denote the generators of the free module on the right hand side and $E_{i,j}^{(r,s)}$ denote the $r \times s$ matrices possessing only one non-zero entry of value 1 at position i, j . The lower index (1) signifies that we only consider the homogeneous part of degree 1 in \underline{s} .

Lemma 3.15. The map Λ induces an isomorphism of first order deformations of $(X_0, 0)$ and (Y_0, E) , i.e. an isomorphism of $\mathbb{C}\{\underline{x}\}$ -modules

$$\Lambda : T_{X_0,0}^1 \xrightarrow{\cong} N'.$$

Proof. We have already obtained the isomorphism Λ between $\text{Mat}(3, 2; \mathbb{C}\{\underline{x}\})$ and $(\mathbb{C}\{\underline{x}\}[s_1, s_2])_{(1)}^3$. From the description of the $T_{X_0,0}^1$ in Lemma 2.1 and the definition of N' we know the relations on both sides. It hence remains to prove that the modules $J_M + \text{Im}(g)$ from Lemma 2.1 and $(J_H + I_H)_{(1)}$ are isomorphic. Here $I_H = \langle H_1, H_2, H_3 \rangle \mathbb{C}\{\underline{x}\}^3$ and J_H is generated by the columns of the Jacobian matrix of the H_i defining Y_0 .

By construction of \underline{H} , we see immediately

$$\begin{aligned} \Lambda\left(\frac{\partial M}{\partial x_i}\right) &= \frac{\partial \underline{H}}{\partial x_i}, \\ \Lambda(M \cdot E_{i,j}^{(2,2)}) &= s_i \frac{\partial \underline{H}}{\partial s_j} \end{aligned}$$

and

$$\Lambda(E_{i,j}^{(3,3)} \cdot M) = H_j e_i.$$

This provides a 1 : 1 correspondence of the generators of these two modules and hence proves the claim about the cokernels:

$$\begin{array}{ccccccc} 0 & \longrightarrow & J_M + \text{Im}(g) & \longrightarrow & \text{Mat}(3, 2; \mathbb{C}\{\underline{x}\}) & \longrightarrow & T_{X_0,0}^1 \longrightarrow 0 \\ & & \downarrow \Lambda \cong & & \downarrow \Lambda \cong & & \downarrow \\ 0 & \longrightarrow & (J_H + I_H)_{(1)} & \longrightarrow & (\mathbb{C}\{\underline{x}\}[s_1, s_2])_{(1)}^3 & \longrightarrow & N' \longrightarrow 0 \end{array}$$

□

There is a splitting of the module N' coming from the local-to-global spectral sequence of the exact sequence of sheaves

$$0 \longrightarrow T_{Y_0} \longrightarrow \iota^* T_{\mathbb{P}^1 \times \mathbb{C}^5} \longrightarrow N_{Y_0} \longrightarrow T_{Y_0}^1 \longrightarrow 0, \quad (9)$$

which can be explicitly described as follows.

We first split the exact sequence (9) into short exact sequences

$$0 \longrightarrow T_{Y_0} \longrightarrow \iota^* T_{\mathbb{P}^1 \times \mathbb{C}^5} \longrightarrow \mathcal{K} \longrightarrow 0 \quad . \quad (10)$$

$$0 \longrightarrow \mathcal{K} \longrightarrow N_{Y_0} \longrightarrow T_{Y_0}^1 \longrightarrow 0$$

The long exact sequences in cohomology both have to finish after the degree one terms, because the underlying scheme is covered by two affine charts.

Let again \mathcal{I} be the ideal sheaf of (Y_0, \mathbb{P}^1) . If we tensor the short exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_{\mathbb{P}^1 \times \mathbb{C}^5} \longrightarrow \mathcal{O}_{Y_0} \longrightarrow 0$$

with the locally free sheaf $T_{\mathbb{P}^1 \times \mathbb{C}^5}$ and take the long exact sequence in cohomology, we see that

$$H^1(Y_0, \iota^* T_{\mathbb{P}^1 \times \mathbb{C}^5}) = 0.$$

Looking at the first long exact sequence in cohomology of (10), we deduce that

$$\text{coker}(H^0(Y_0, \iota^* T_{\mathbb{P}^1 \times \mathbb{C}^5}) \rightarrow H^0(Y_0, \mathcal{K})) \cong H^1(Y_0, T_{Y_0}) \quad (11)$$

and

$$H^1(Y_0, \mathcal{K}) = 0. \quad (12)$$

Combining these results with the second long exact sequence of (10) and recalling that $N' = N_{Y_0}/H^0(Y_0, \iota^* T_{\mathbb{P}^1 \times \mathbb{C}^5})$, we obtain a short exact sequence

$$0 \longrightarrow H^1(Y_0, T_{Y_0}) \longrightarrow N' \longrightarrow H^0(Y_0, T_{Y_0}^1) \longrightarrow 0 ,$$

the middle term of which is a finite dimensional vector space over \mathbb{C} . Any choice of a splitting gives us

$$N' = H^1(Y_0, T_{Y_0}) \oplus H^0(Y_0, T_{Y_0}^1). \quad (13)$$

The sheaf underlying the right hand side summand is supported only in the singular points and hence affine. Thus if we let $\Sigma(Y_0)$ be the set of singular points of Y_0 we can rewrite (13) as

$$N' = H^1(Y_0, T_{Y_0}) \oplus \bigoplus_{p \in \Sigma(Y_0)} T_{Y_0, p}^1 \quad (14)$$

In particular for any $q \in \Sigma(Y_0)$ we get a surjective map from $T_{X_0, 0}^1$ onto $T_{Y_0, q}^1$ by the composition

$$T_{X_0, 0}^1 \cong N' \cong H^1(Y_0, T_{Y_0}) \oplus \bigoplus_{p \in \Sigma(Y_0)} T_{Y_0, p}^1 \longrightarrow T_{Y_0, q}^1,$$

where the last map is the projection to the summand for q . This proves the following corollary.

Corollary 3.16. Let $(X_0, 0) \subset (\mathbb{C}^5, 0)$ be an ICMC2 threefold singularity of Cohen-Macaulay type $t = 2$ such that the Tjurina transform Y_0 has at most isolated singularities. Furthermore let $X_0 \hookrightarrow X \longrightarrow \mathbb{C}^\tau$ be a semi-universal deformation of X_0 . Then the induced family $Y_0 \hookrightarrow Y \longrightarrow \mathbb{C}^\tau$ is again versal for each of the arising singularities.

Note that the induced local deformations for the isolated singularities of Y_0 do not need to be semi-universal, i.e. τ might not be minimal.

Remark 3.17. As can be expected given the results of this section, for all $(X_0, 0)$ in the table of simple ICMC2 singularities of dimension 3, the Tjurina modification Y_0 has at most simple ICIS as can be read off from table 1.

4 Vanishing cycles

From now on we'll often be concerned with the homology groups of a given topological space. By this we mean Simplicial homology with integer coefficients and we'll just write $H_\bullet(-)$ for $H_\bullet(-, \mathbb{Z})$ for short. To fix notation, we briefly recall the definition of Milnor fiber and vanishing cycles for isolated singularities (see e.g. [20] for a reference on these topics).

Let $(X_0, 0) \subset (\mathbb{C}^n, 0)$ be an isolated singularity at the origin and X_0 a representative thereof. Then there is a real $\eta_0 > 0$ such that the intersection of X_0 with the sphere S_η of radius η is transversal for all $\eta_0 \geq \eta > 0$. For any $\eta > 0$ chosen in this way, we will refer to a closed ball B_η of radius η around $0 \in \mathbb{C}^n$ as a Milnor ball for the singularity X_0 . Furthermore we denote by $\overline{X_0}$ the topological space

$$\overline{X_0} := X_0 \cap B_\eta.$$

We explicitly cite the following well-known theorem, which ensures that these definitions are independent of the chosen (sufficiently small) η .

Theorem 4.1 (conical structure, [20]). For an isolated singularity $(X_0, 0) \subset (\mathbb{C}^n, 0)$ and a Milnor ball B_η for X_0 the pair of spaces (B_η, \overline{X}_0) is homeomorphic to the pair $(C(S_\eta), C(\partial\overline{X}_0))$, where $C(L)$ denotes the cone over $L \subset S_\eta$, i.e. the set of real line segments to the origin. This can be chosen to be a diffeomorphism on the open set $(B_\eta \setminus \{0\}, \overline{X}_0 \setminus \{0\})$.

Consequently \overline{X}_0 and $\partial\overline{X}_0$ are well defined topological spaces up to homeomorphism for a germ $(X_0, 0)$ of an isolated singularity, i.e. do not depend on the representative X_0 . Now, consider a deformation of an isolated singularity $(X_0, 0) \subset (\mathbb{C}^n, 0)$ by some parameter ε , i.e. a flat family

$$\begin{array}{ccc} X_0 & \xrightarrow{\iota} & X \\ \downarrow & & \downarrow \varepsilon \\ \{0\} & \longrightarrow & \mathbb{C} \end{array}$$

where $X_0 \subset \mathbb{C}^n$ and $X \subset \mathbb{C}^n \times \mathbb{C}$ are representatives of the respective germs. Having chosen a Milnor ball B_η for X_0 there exists an open neighborhood $0 \in D \subset \mathbb{C}$ in the deformation base \mathbb{C} , such that for all $\varepsilon \in D$ the intersection

$$\partial\overline{X}_\varepsilon = X_\varepsilon \cap \partial B_\varepsilon$$

of the fiber X_ε with the Milnor ball in the fiber over ε is transversal. The cylinder $B_\eta \times D$ is called a Milnor tube for the deformation of X_0 .

Theorem 4.1 ensures that all the homology groups of \overline{X}_0 vanish except in degree 0. But in any deformed fiber \overline{X}_ε there may exist nontrivial cycles. If a fiber \overline{X}_ε is smooth, i.e. a smooth complex manifold with boundary, it is called a Milnor fiber of the singularity X_0 . Any nontrivial cycles in the homology of \overline{X}_ε are called vanishing cycles of the singularity X_0 . It is well known what these vanishing cycles look like for any ICIS of complex dimension n : they form a bouquet of spheres of real dimension n , see [17]

For the ICMC2 singularities of dimension 3 which we are considering in this article, known results on the vanishing cycles of the Milnor fibers are scarce.

Remark 4.2. It is a priori not clear and in general wrong to expect exactly one Milnor fiber for a given isolated singularity X_0 (up to diffeomorphism). First of all there may not exist any deformation with smooth fibers at all, like for the rigid isolated 4-fold singularity appearing in the classification of Fröhbis-Krüger and Neumer.

On the other hand, given two different smoothings $\pi : X \rightarrow \mathbb{C}$ and $\pi' : X' \rightarrow \mathbb{C}$ two smooth fibers X_ε and X'_ε are not necessarily diffeomorphic as the famous example of Pinkham [23] shows. They are, however, diffeomorphic if they belong to the same connected component of the deformation base. This is an immediate corollary of the Ehresmann fibration theorem.

If X_0 is a smoothable ICMC2 singularity, the set of points $\varepsilon \in \mathbb{C}^\tau$ with smooth fibers is open and connected since its complement (the discriminant) has real codimension at least 2. Therefore in all our cases of interest in this article the singularities have a unique Milnor fiber up to diffeomorphism.

We need one more preliminary result which will be applied to determine the topology of the Tjurina modification Y_0 .

Proposition 4.3. Let $(X_0, 0) \subset (\mathbb{C}^n, 0)$ be an isolated singularity and $\pi_0 : Y_0 \rightarrow X_0$ a morphism defined on suitably small representatives such that the restriction

$$\pi_0 : Y_0 \setminus \pi_0^{-1}(\{0\}) \rightarrow X_0 \setminus \{0\}$$

is an isomorphism and the exceptional set $E = \pi_0^{-1}(\{0\})$ is closed and projective. Then E is a deformation retract of Y_0 .

Proof. The variety E is closed and projective, hence compact. It follows from [18], that E is a Euclidean Neighborhood Retract of an open neighborhood U of E in Y_0 . But outside E the map π_0 is an isomorphism, so $\pi_0(U) \subset X_0$ is open. With the theorem about the conical structure (4.1) we can now shrink $\bar{Y}_0 \setminus E = \bar{X}_0 \setminus \{0\}$ to something homotopic to \bar{Y}_0 inside the open set $\pi_0(U)$ and subsequently to E . \square

Using Tjurina modification in family, we are now ready to explain the observations of [9] in the case of a simple ICMC2 threefold $(X_0, 0) \subset (\mathbb{C}^5, 0)$. Applying the Tjurina modification we get a transform Y_0 with only A-D-E singularities according to Remark 3.17. Since we necessarily have Cohen-Macaulay type $t = 2$, the homotopy type of Y_0 is given by the exceptional set $\mathbb{P}^1 = E \subset Y_0$ as a consequence of Proposition 4.3. So we always find

$$b_0(Y_0) = 1, \quad b_1(Y_0) = 0, \quad b_2(Y_0) = 1, \quad b_3(Y_0) = 0. \quad (15)$$

Now Proposition 3.9 assures the Tjurina modification to be well behaved within families. Hence we can choose any smoothing $X_0 \hookrightarrow X \xrightarrow{\varepsilon} \mathbb{C}$ and carefully observe the interplay of cycles present in Y_0 with upcoming vanishing cycles of the ICIS when passing from Y_0 to a deformed fiber Y_ε in the induced deformation. This is covered in Theorem 4.4. Finally we can use the identification $Y_\varepsilon \cong X_\varepsilon$ from Proposition 3.8 to obtain the desired vanishing topology.

We slightly weaken the assumptions and also allow $(X_0, 0) \subset (\mathbb{C}^5, 0)$ to be a non-simple ICMC2 threefold singularity. However we still require the Cohen-Macaulay type to be $t = 2$ and the jet type as in Lemma 3.14, i.e. only ICIS in Y_0 ; the more general case allowing non-isolated singularities in the Tjurina transform will be studied in [28] in more detail. The results are gathered in the Tables 1 and 2 in the next section.

Theorem 4.4. In the above setting consider X_0 as the special fiber in a smoothing $X_0 \hookrightarrow X \xrightarrow{\varepsilon} \mathbb{C}$ together with the Tjurina modification in family $Y_0 \hookrightarrow Y \xrightarrow{\varepsilon \circ \pi} \mathbb{C}$ as in diagram 5. We denote the Milnor tube arising from B by

$$T = B \times D \subset \mathbb{C}^5 \times \mathbb{C}$$

and the one originating from \hat{B} by

$$\hat{T} = \pi^{-1}(T) \subset \mathbb{C}^5 \times \mathbb{C} \times \mathbb{P}^1,$$

which also allows us to refer to the Milnor fiber \overline{X}_ε and the fiber $\overline{Y}_\varepsilon = \hat{T} \cap Y_\varepsilon$ sitting over it. The Betti numbers of a smooth fiber \overline{X}_ε are given by

$$b_0 = 1, \quad b_1 = 0, \quad b_2 = 1, \quad b_3 = r$$

where $r \in \mathbb{N}$ is the sum of the Milnor numbers of the ICIS of Y_0 .

Proof. Throughout the proof many steps require shrinking the open set of admissible deformation parameters ε in the deformation base. However, those steps are finitely many and no harm is done, since we only consider representatives of germs. For the reader's convenience, we will suppress mentioning this obvious technical detail each time it occurs.

Let n be the number of singularities of \overline{Y}_0 . Fix local analytic embeddings of these ICIS to some affine space and let $T = \bigcup_{i=1}^n T_i$ be a collection of Milnor tubes around the singularities of \overline{Y}_0 for the induced deformation $\overline{Y}_0 \hookrightarrow \overline{Y} \xrightarrow{\varepsilon \circ \pi} \mathbb{C}$. For each $i = 1, \dots, n$, let S_i be the Milnor tubes swept out by Milnor balls of half of the radius of T_i . Decompose the total space \overline{Y} in the two open sets

$$U := \overline{Y} \setminus \left(\bigcup_{i=1}^n S_i \cap Y \right) \quad \text{and} \quad V \text{ the interior of } \left(\bigcup_{i=1}^n T_i \cap Y \right)$$

Let $\overline{U}, \overline{V}$ and $\overline{W} = \overline{U \cap V}$ be the closures of the respective open sets in the Euclidean topology. Each of them is compact, \overline{U} and \overline{W} even compact manifolds with boundary. An illustration of this setting can be found in Figure 1.

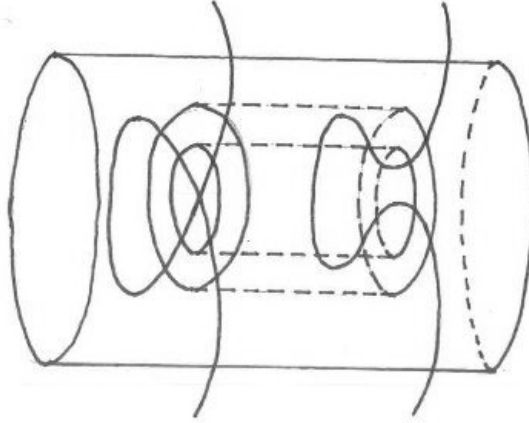


Figure 1: \overline{Y} and the Milnor tubes S and T

By the Ehresmann fibration theorem we can choose a differentiable flow

$$\Phi : D \times U' \rightarrow U'$$

defined on an open set $0 \in D \subset \mathbb{C}$ in the deformation base and a neighborhood U' of the closure \overline{U} in \overline{Y} , such that the restriction to \overline{U}_0

$$\Phi|_{\{\varepsilon\} \times \overline{U}_0} : \overline{U}_0 \rightarrow \overline{U}_\varepsilon$$

and the restriction to the overlap

$$\Phi|_{\{\varepsilon\} \times \overline{W}_0} : \overline{W}_0 \rightarrow \overline{W}_\varepsilon.$$

are diffeomorphisms of manifolds with boundary for sufficiently small $|\varepsilon|$. Now consider the long exact sequence in reduced homology for the pair of spaces $(\overline{Y}_0, \overline{V}_0)$:

$$\dots \longrightarrow H_i(\overline{V}_0) \longrightarrow H_i(\overline{Y}_0) \longrightarrow H_i(\overline{Y}_0, \overline{V}_0) \longrightarrow H_{i-1}(\overline{V}_0) \longrightarrow \dots \quad (16)$$

Because \overline{V}_0 is contractible to a union of points, we find $H_i(\overline{Y}_0) \rightarrow H_i(\overline{Y}_0, \overline{V}_0)$ to be an isomorphism for $i > 0$. Clearly,

$$H_i(\overline{Y}_0, \overline{V}_0) = H_i(\overline{Y}_0/\overline{V}_0) = H_i(\overline{U}_0/\partial\overline{U}_0) = H_i(\overline{U}_\varepsilon/\partial\overline{U}_\varepsilon) = H_i(\overline{Y}_\varepsilon, \overline{V}_\varepsilon), \quad (17)$$

because of excision. The identifications in equation 17 and equation 15 provide us with zeros for the terms $H_i(\overline{Y}_\varepsilon, \overline{V}_\varepsilon) = 0$ for $i \neq 0, 2$. Now consider the analogous long exact sequence for the pair of spaces $(\overline{Y}_\varepsilon, \overline{V}_\varepsilon)$ and obtain in degree 3

$$0 \longrightarrow H_3(\overline{V}_\varepsilon) \longrightarrow H_3(\overline{Y}_\varepsilon) \longrightarrow 0 \quad (18)$$

which means that every vanishing cycle of the occurring singularities in Y_0 is preserved in the whole fiber X_ε . Since for ICIS singularities the Milnor fiber has the topological type of a bouquet of μ 3-spheres and we have a decomposition

$$H_3(\overline{V}_\varepsilon) = \bigoplus_{i=1}^n H_3(\overline{V}_{i,\varepsilon}),$$

the middle Betti number b_3 of \overline{Y}_ε is the sum of Milnor numbers of the singularities of Y_0 .

The Tjurina transform Y_0 has only ICIS, so we get zeros for $H_i(\overline{V}_\varepsilon)$ for $i \neq 0, 3$ which leads to

$$0 \longrightarrow H_2(\overline{Y}_\varepsilon) \longrightarrow H_2(\overline{Y}_\varepsilon, \overline{V}_\varepsilon) \longrightarrow 0 \quad (19)$$

in degree 2. As the fibers \overline{Y}_ε and \overline{X}_ε are isomorphic according to Proposition 3.8, the claim is proved. \square

Example 4.5. The applied ideas also work for higher Cohen-Macaulay type, as we would like to illustrate by revisiting our previous Example 3.7 from Section 3: also in the case $t = 3$ Proposition 3.9 assures the Tjurina modification to work in family. Contrary to the case $t = 2$, the central fiber in the Tjurina modification Y_0 now has the homotopy type of $E = \pi_0^{-1}(\{0\}) = \mathbb{P}^2$. Thus the Betti numbers read

$$b_0(Y_0) = 1, \quad b_1(Y_0) = 0, \quad b_2(Y_0) = 1, \quad b_3(Y_0) = 0, \quad b_4(Y_0) = 1. \quad (20)$$

Clearly the 4-cycle generated by the \mathbb{P}^2 itself can not be preserved in a smooth fiber $Y_\varepsilon \cong X_\varepsilon$, because X_ε is affine and we would get a contradiction to the Lefschetz Hyperplane Theorem. In fact this cycle breaks at the 10 points of the A_1 singularities of Y_0 when resolving them. We can again calculate along the lines of the proof of Theorem 4.4 using the long exact sequence of pairs of spaces. The degree 2 part stays isolated, so we again get

$$b_2(X_\varepsilon) = 1.$$

For degree 3 the calculations show, that the broken \mathbb{P}^2 leads to a relation among the vanishing cycles of the A_1 's. Hence we have

$$b_3(X_\varepsilon) = 9$$

and not 10 as one might have expected.

Remark 4.6. A similar phenomenon can be observed when projectivizing the column space of the matrix M of an ICMC2 threefold $(X_0, 0)$ of Cohen-Macaulay type $t = 2$. As in the case of a Tjurina modification, projectivizing the column space is compatible with deformations, because we again get a locally complete intersection transform, say Z_0 . Only this time we find a \mathbb{P}^2 as exceptional set. If we apply this to the first entry of the table 1, also called the A_0^+ singularity, we find

$$b_0(Z_0) = 1, \quad b_1(Z_0) = 0, \quad b_2(Z_0) = 1, \quad b_3(Z_0) = 0, \quad b_4(Z_0) = 1$$

and one A_1 singularity in Z_0 . Again, the induced deformation destroys the 4-cycle of Z_0 leading to the vanishing cycle of the A_1 singularity being homologous to zero in $X_\varepsilon = Z_\varepsilon$.

5 The topological type of the simple ICMC2 singularities

Using the results of the previous section, direct computation now provides explicit results for the structure of the Milnor fiber for the simple ICMC2 singularities and for the bounding non-simple ones. We have summarized the results in the following two tables. Subsequently, we finish this article by pointing out and explaining some notable observations and stating some arising questions.

M^T	τ	sing. in Y_0	b_2	b_3
$\begin{pmatrix} x & y & z \\ v & w & x \end{pmatrix}$	1	-	1	0
$\begin{pmatrix} x & y & z \\ v & w & x^{k+1} + y^2 \end{pmatrix}$	$k + 2$	A_k	1	k
$\begin{pmatrix} x & y & z \\ v & w & xy^2 + x^{k-1} \end{pmatrix}$	$k + 2$	D_k	1	k
$\begin{pmatrix} x & y & z \\ v & w & x^3 + y^4 \end{pmatrix}$	8	E_6	1	6
$\begin{pmatrix} x & y & z \\ v & w & x^3 + xy^3 \end{pmatrix}$	9	E_7	1	7
$\begin{pmatrix} x & y & z \\ v & w & x^3 + y^5 \end{pmatrix}$	10	E_8	1	8
$\begin{pmatrix} w & y & x \\ z & w & y + v^k \end{pmatrix}$	$2k - 1$	-	1	0
$\begin{pmatrix} w & y & x \\ z & w & y^k + v^2 \end{pmatrix}$	$k + 2$	A_{k-1}	1	$k - 1$
$\begin{pmatrix} w & y & x \\ z & w & yv + v^k \end{pmatrix}$	$2k$	A_1	1	1
$\begin{pmatrix} w + v^k & y & x \\ z & w & yv \end{pmatrix}$	$2k + 1$	A_1	1	1
$\begin{pmatrix} w + v^2 & y & x \\ z & w & y^2 + v^k \end{pmatrix}$	$k + 3$	A_{k-1}	1	$k - 1$
$\begin{pmatrix} w & y & x \\ z & w & y^2 + v^3 \end{pmatrix}$	7	A_2	1	2
$\begin{pmatrix} v^2 + w^k & y & x \\ z & w & v^2 + y^l \end{pmatrix}$	$k + l + 1$	A_{k-1}, A_{l-1}	1	$k + l - 2$
$\begin{pmatrix} v^2 + w^k & y & x \\ z & w & yv \end{pmatrix}$	$k + 4$	A_{k-1}, A_1	1	k
$\begin{pmatrix} v^2 + w^k & y & x \\ z & w & y^2 + v^l \end{pmatrix}$	$k + l + 2$	A_{k-1}, A_{l-1}	1	$k + l - 2$
$\begin{pmatrix} wv + v^k & y & x \\ z & w & yv + v^k \end{pmatrix}$	$2k + 1$	A_1, A_1	1	2
$\begin{pmatrix} wv + v^k & y & x \\ z & w & yv \end{pmatrix}$	$2k + 2$	A_1, A_1	1	2

$\begin{pmatrix} wv + v^3 & y & x \\ z & w & y^2 + v^3 \end{pmatrix}$	8	A_1, A_2	1	3
$\begin{pmatrix} wv & y & x \\ z & w & y^2 + v^3 \end{pmatrix}$	9	A_1, A_2	1	3
$\begin{pmatrix} w^2 + v^3 & y & x \\ z & w & y^2 + v^3 \end{pmatrix}$	9	A_2, A_2	1	4
$\begin{pmatrix} z & y & x \\ x & w & v^2 + y^2 + z^k \end{pmatrix}$	$k + 4$	D_{k+1}	1	$k + 1$
$\begin{pmatrix} z & y & x \\ x & w & v^2 + yz + y^k w \end{pmatrix}$	$2k + 5$	A_{2k+2}	1	$2k + 2$
$\begin{pmatrix} z & y & x \\ x & w & v^2 + yz + y^{k+1} \end{pmatrix}$	$2k + 4$	A_{2k+1}	1	$2k + 1$
$\begin{pmatrix} z & y & x \\ x & w & v^2 + yw + z^2 \end{pmatrix}$	8	D_5	1	5
$\begin{pmatrix} z & y & x \\ x & w & v^2 + y^3 + z^2 \end{pmatrix}$	9	E_6	1	6
$\begin{pmatrix} z & y & x + v^2 \\ x & w & vy + z^2 \end{pmatrix}$	7	D_3	1	3
$\begin{pmatrix} z & y & x + v^2 \\ x & w & vz + y^2 \end{pmatrix}$	8	A_4	1	4
$\begin{pmatrix} z & y & x + v^2 \\ x & w & z^2 + y^2 \end{pmatrix}$	9	D_5	1	5

Table 1: Homology of Milnor fibers computed by means of the Tjurina modification

M^T	τ	sing. in Y_0	b_2	b_3
$\begin{pmatrix} x & y & z \\ w & v & x^4 + y^4 \end{pmatrix}$	11	X_9	1	9
$\begin{pmatrix} x & y & z \\ w & v & x^3 + y^6 \end{pmatrix}$	12	J_{10}	1	10
$\begin{pmatrix} w + v^2 & y & x \\ z & v & y^3 + v^3 \end{pmatrix}$	8	D_4	1	4
$\begin{pmatrix} w + v^3 & y & x \\ z & w & y^2 + v^4 \end{pmatrix}$	9	A_3	1	3
$\begin{pmatrix} z & y & x \\ x & w & v^2 + y^3 + z^3 \end{pmatrix}$	11	$T_{3,3,3}$	1	8
$\begin{pmatrix} z & y & x \\ x & w & v^3 + y^2 + z^3 \end{pmatrix}^5$	13	$T_{3,3,3}$	1	8

⁵There is a typesetting error in this matrix in [10]. The right-hand lower entry here is the correct one.

$\begin{pmatrix} z & y & x \\ x & w & v^3 + y^3 + z^2 \end{pmatrix}$	17	U_{12}	1	12
$\begin{pmatrix} z & y & x \\ x & w & v^2 + y^4 + z^2 \end{pmatrix}$	12	X_9	1	9
$\begin{pmatrix} z & y & x + v^2 \\ x & w & vz + yz + vw \end{pmatrix}$	10	D_6	1	6
$\begin{pmatrix} z & y & x + v^3 \\ x & w & vy + z^2 \end{pmatrix}$	9	A_3	1	3
$\begin{pmatrix} z & y & x + v^3 \\ x & w & y^2 + yz + z^2 \end{pmatrix}$	15	X_9	1	9
$\begin{pmatrix} z & y & x + v^2 \\ x & w & vy + yz + z^3 \end{pmatrix}$	8	D_4	1	4

Table 2: Homology of Milnor fibers for the bounding non-simple singularities

Remark 5.1. (direct observations from the table)

1. We only see simple singularities occurring in Y_0 in table (1). In table (2), where the listed singularities are non-simple, there are some simple and some non-simple singularities arising from Tjurina modification. In particular, the non-simple ones arise in the cases with 1-jet types $J^{(5,2)}$ and in some subcases of $J^{(4,4)}$ in the notation of [10], whereas the simple ones occur for $J^{(4,2)}$ and the remaining subcases for $J^{(4,4)}$. The 1-jet types $J^{(4,5)}$ and $J^{(4,6)}$ have not been included in the table, because they lead to non-isolated singularities in Y_0 , the singular locus being the whole exceptional \mathbb{P}^1 .
2. Looking at the preceding tables, one fact immediately attracts attention: For any of the explicitly computed ICMC2 threefold singularities the second Betti number is always 1. The mechanism behind this fact can be explained as follows: According to the identifications (17) and (19), the second homology group of the smooth fiber is inherited from the exceptional set in the Tjurina transform, which is a $\mathbb{P}^{(t-1)}$ depending only on the Cohen-Macaulay type $t = 2$. In particular this vanishing cycle can not be directly related to any deformation parameters in the sense of e.g. the Lê-Greuel formulas.

Our results answer negatively to a question of J. Damon and B. Pike who expected both Betti numbers b_2 and b_3 to grow at the same rate in families of ICMC2 with

$$\mu = b_3 - b_2$$

constant. But our computations show that there are infinite families in which neither of the two changes. The first occurrence of this phenomenon can be

found in line 7 of the preceding table (1), given by the matrix

$$\begin{pmatrix} w & y & x \\ z & w & y + v^k \end{pmatrix}$$

Following the notation of [10], we call these singularities the Π_k family. The topological type stays the same within this family, whereas the isomorphism classes of the space germs do not coincide for different k .

The Tjurina transform Y_0 of the Π_k singularities is smooth. The second summand of the splitting (7) of the space of first order deformations of Y_0 vanishes and we obtain

$$N' = H^1(Y_0, T_{Y_0}).$$

More general if we allow isolated singularities in the Tjurina transform as in Lemma (3.14), the dimension $h^1(Y_0, T_{Y_0})$ measures how many degrees of freedom for deformations of $(X_0, 0)$ become locally trivial for all the singular points in Y_0 .

For all the ICIS at the points in the singular locus $\Sigma(Y_0)$ of the Tjurina transform Y_0 , we get local vanishing cycles in degree 3 as we pass to the smooth fiber. The computations (18) in the proof of Theorem (4.4) show that there can be no global relations between them and that they generate the third homology group of the global Milnor fiber Y_ϵ .

Contrary to the second homology group, in degree three we can use the Lê-Greuel formulas to relate the third Betti number

$$b_3 = \sum_{p \in \Sigma(Y_0)} \mu_p$$

to those of the local singularities (Y_0, p) in the Tjurina transform and to their Tjurina numbers $\tau_p = \dim T_{Y_0, p}^1$. In case of hypersurface singularities in Y_0 , the numbers τ_p and μ_p can be computed as the vector space dimensions of the Tjurina algebra and the Milnor algebra respectively. This covers the case of all simple ICMC2 singularities in $(\mathbb{C}^5, 0)$, because due to Corollary (3.16) we can only find simple 3-dimensional ICIS in Y_0 ; according to Giusti's list of simple ICIS, these can only be A-D-E singularities, see [14]. Since the Milnor and the Tjurina numbers coincide for A-D-E singularities (and more generally for quasihomogeneous hypersurface singularities), we have proved the following theorem.

Theorem 5.2. For ICMC2 threefold singularities of type 2, whose Tjurina transform Y_0 has as singular locus $\Sigma(Y_0)$ either the empty set or a set of dimension 0, we have

$$\tau = h^1(Y_0, T_{Y_0}) + \sum_{p \in \Sigma(Y_0)} \tau_p, \quad (21)$$

which becomes

$$\tau = h^1(Y_0, T_{Y_0}) + b_3. \quad (22)$$

for the simple ICMC2 singularities.

There seems to be a relation between the number $h^1(Y_0, T_{Y_0})$ and the maximal number of matrix singularities to which $(X_0, 0)$ can be deformed. The final object in the adjacencies among ICMC2 3-fold singularities of Cohen-Macaulay type 2 is the so called A_0^+ singularity

$$\begin{pmatrix} x & y & z \\ v & w & x \end{pmatrix},$$

which is the first entry of the table (1). Consider the Π_k -family from table 1 and the deformation over \mathbb{C} given by

$$\begin{pmatrix} w & y & x \\ z & w & y + v^k \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}.$$

For $\varepsilon \neq 0$ we find k distinct singularities at the points $(x, y, z, v, w) = (0, 0, 0, \sqrt[k]{\varepsilon}, 0)$. Using the analytic coordinate change $v' = y + v^k - \varepsilon$ locally at any of these points gives the standard form of the A_0^+ . This and other examples yield $h^1(Y_0, T_{Y_0})$ to grow linearly with the maximal number of A_0^+ singularities on a neighboring singular fiber, an observation which coincides with the fact that the Tjurina transform is blind to components of the discriminant above which only determinantal singularities exist. This train of thought is pursued in detail in [12].

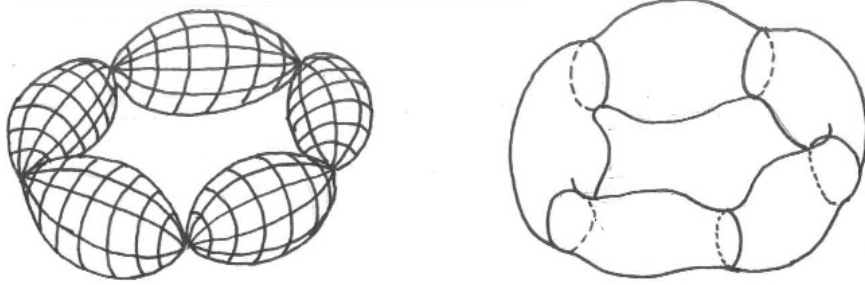


Figure 2: The wrapped candy chain for $k = 5$ and $\varepsilon \neq 0$ before and after deforming the A_0^+ singularities

The preceding example also exhibits another interesting behaviour in the topology. While the Milnor fiber has its third homology group equal to zero, the fiber over $\varepsilon \neq 0$ contains a bouquet of k real 3-spheres as indicated in Figure 2. It reminds us of a chain of wrapped candy. It turns out that their total sum is homologous to zero, while either $k - 1$ of them generate $H_3(X_\varepsilon)$. The A_0^+ singularities sit at the “wrapping points” between these spheres and are unraveled when passing to a smooth fiber. Consequently all local 2-cycles become pairwise homologous. A direct computation then shows that the appearing 3-chain is in fact homologous to 0.

References

- [1] V. Arnold. “Normal forms for functions near degenerate critical point”. In: *FAA* 6 (1972), pp. 254–272.
- [2] M. Artin. *Lectures on Deformations of Singularities*. Tata Institute of Fundamental Research, Bombay, 1976.
- [3] W. Bruns and U. Vetter. *Determinantal Rings*. Vol. 1327. Lecture Notes in Mathematics. Springer, 1988.
- [4] R. Buchweitz. “Contributions à la théorie des singularités”. Thèse. Paris, 1981.
- [5] L. Burch. “On ideals of finite homological dimension in local rings”. In: *Proc. Camb. Phil. Soc.* 64 (1964), pp. 941–948.
- [6] N. Chachapoyas Siesquen. “Invariantes de variedades determinantaís”. PhD-Thesis. Universidade de Sao Paulo, Sao Carlos, 2014.
- [7] J. Damon. “Deformations of Sections of Singularities and Gorenstein Surface Singularities”. In: *Amer. J. Math.* 109 (1987), pp. 695–721.
- [8] J. Damon. “Nonlinear sections of nonisolated complete intersections”. In: *New Developments in Singularity Theory*. Vol. 21. NATO Conf. Series. Kluwer, 2001, pp. 405–445.
- [9] J. Damon and B. Pike. “Solvable groups, free divisors and nonisolated matrix singularities II: Vanishing topology”. In: *Geom. Topol.* 18 (2014), pp. 911–962.
- [10] A. Fruehbs-Krueger and A. Neumer. “Simple Cohen-Macaulay Codimension 2 Singularities”. In: *Comm. in Alg.* 38.2 (2010), pp. 454–495.
- [11] A. Frühbis-Krüger. “Classification of simple space curve singularities”. In: *Comm. Algebra* 27.8 (1999), pp. 3993–4013.
- [12] A. Frühbis-Krüger. “On discriminants, Tjurina modifications and the geometry of determinantal singularities”. In: (2016). eprint: [arXiv:1611.02625](#).
- [13] T. Gaffney and A. Rangachev. “Pairs of Modules and Determinantal Isolated Singularities”. In: (2015). eprint: [arXiv:1501.00201](#).
- [14] M. Giusti. “Classification des singularités isolées simples d’intersections complètes”. In: *Proc. Symp Pure Math.* 40 (1983), pp. 457–494.
- [15] G.-M. Greuel. “On deformation of curves and a formula of Deligne”. In: *Algebraic Geometry, La Rabida 1981*. Vol. 961. Lecture Notes in Math. Springer, 1983, pp. 141–168.
- [16] G.-M. Greuel and J. Steenbrink. “On the topology of smoothable singularities”. In: *Singularities, Part 1 (Arcata, Calif., 1981)*. Vol. 40. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, R.I., 1983, pp. 535–545.

- [17] H. Hamm. “Lokale topologische Eigenschaften komplexer Räume”. In: *Math. Ann.* 191 (1972), pp. 235–252.
- [18] S. Lojasiewicz. “Triangularities of semi analytic sets”. In: *Ann. Scuola Norm. Sup. Pisa (3)* 18 (1964), pp. 449–474.
- [19] E. Looijenga and G.-M. Greuel. “Milnor number and Tjurina number of Complete Intersections”. In: *Math. Ann.* 271 (1985), pp. 121–124.
- [20] J. Milnor. *Singular points of complex hypersurfaces*. Princeton University Press, 1968.
- [21] J.W. Milnor. *Morse Theory*. Princeton, NJ: Princeton University Press, 1963.
- [22] J.J. Nuno-Ballesteros, B. Oréfiçe-Okamoto, and J.N. Tomazella. “The vanishing Euler characteristic of an isolated determinantal singularity”. In: *Israel J. Math.* 197 (2013), pp. 475–495.
- [23] H. Pinkham. “Deformations of algebraic varieties with G_m action”. In: vol. 20. *Astérisque*. Soc. Math. France, 1974.
- [24] M. Schaps. “Deformations of Cohen-Macaulay schemes of codimension 2 and nonsingular deformations of space curves”. In: *Amer. J. Math.* 99 (1977), pp. 669–684.
- [25] M. da Silva Pereira and M. Soares Ruas. “Codimension two Determinantal Varieties with Isolated Singularities”. In: *Math.Scand.* 115 (2014), pp. 161–172.
- [26] D. van Straten. “Weakly Normal Surface Singularities and Their Improvements”. PhD-Thesis. Universiteit Leiden, 1987.
- [27] G. N. Tjurina. “Absolute isolatedness of rational singularities and triple rational points”. In: *Func. Anal. Appl.* 2 (1968), pp. 324–333.
- [28] M. Zach. “Vanishing Cycles of smoothable Isolated Cohen-Macaulay codimension 2 singularities of type 2”. In: (2016). eprint: [arXiv:1607.07527](https://arxiv.org/abs/1607.07527).